

Hypergraph-theoretic characterizations for LOCC incomparable ensembles of multipartite CAT states*

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Using graphs and hypergraphs to systematically model collections of arbitrary subsets of parties representing *ensembles (or collections)* of shared multipartite CAT states, we study transformations between such *ensembles* under *local operations and classical communication (LOCC)*. We show using partial entropic criteria, that any two such distinct ensembles represented by *r-uniform hypergraphs* with the same number of hyperedges (CAT states), are LOCC incomparable for even integers $r \geq 2$, generalizing results in [10, 12]. We show that the cardinality of the largest set of mutually LOCC incomparable ensembles represented by *r-uniform hypergraphs* for even $r \geq 2$, is exponential in the number of parties. We also demonstrate LOCC incomparability between two ensembles represented by 3-uniform hypergraphs where partial entropic criteria do not help in establishing incomparability. Further we characterize LOCC comparability of EPR graphs in a model where LOCC is restricted to teleportation and edge destruction. We show that this model is equivalent to one in which LOCC transformations are carried out through a sequence of operations where each operation adds at most one new EPR pair.

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I. INTRODUCTION

Certain operations like entanglement teleportation and creation of multipartite entanglement states can be done using only classical communication, with the aid of preshared quantum entanglement between the geographically separated parties [1, 2]. Correlations in different problems being solved between various subsets of parties in a scalable network may be exploited by using multiple preshared entanglements within those subsets of parties for reducing classical communication complexity. See [6, 7, 8] for problems where such savings are possible. The specifications of requisite patterns of entanglement may change over a period of time in a quantum computation network; we may require to solve different problems between different sets of combinations of parties. In such a scenario, it becomes necessary to transform one set of entanglement combinations across the network, into another distinctly different set of entanglements. The main question is whether such transformations from one pattern of multiple preshared entanglements between parties to another such pattern are possible using only LOCC (*local operations and classical communication*). Nielsen [2, 3] derived important results about conditions for LOCC transformations between bipartite states and the partial order between such states. Linden et al. [9], considered reversible transformations using local quantum operations and classical communication for multi-particle environments. For multipartite entanglement ensembles of CAT states shared between various combinations of parties, several important characterizations of LOCC incomparability were derived in [10, 12] using the method of *bicolored merging*, based on *partial entropic criteria*. In this work we further characterize and classify certain incomparable ensembles of multipartite CAT states combinatorially, and study LOCC transformations between ensembles that are not incomparable. We study the scope and limitations of partial entropic criteria in establishing LOCC incomparability between multipartite states. Before

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we outline our contribution, we present a few necessary definitions and some notation.

We need a few definitions and some notation. An *EPR graph* $G(V, E)$ is a graph whose vertex set V is a set of *parties*, and an edge $\{u, v\}$, where $u, v \in V$, represents shared entanglement in the form of an EPR pair between the parties u and v . An *entanglement configuration hypergraph* (*EC hypergraph*) $H(S, F)$, has a set S of n parties and a set $F = \{E_1, E_2, \dots, E_m\}$ of m hyperedges, where $E_i \subseteq S; i = 1, 2, \dots, m$, and E_i is such that its elements (parties) share an $|E_i|$ -CAT state. So, an EPR graph or an EC hypergraph represents multipartite states with multiple entanglements in the form of CAT states, where each CAT state is represented by an edge or hyperedge, respectively. If one such multipartite state $|\phi\rangle$ can be transformed into another such state $|\psi\rangle$ by LOCC, then we denote this transformation as $|\phi\rangle \geq |\psi\rangle$ (or $|\psi\rangle \leq |\phi\rangle$). If none of $|\phi\rangle \geq |\psi\rangle$ and $|\psi\rangle \geq |\phi\rangle$ hold then we say that the two ensembles or states are *LOCC incomparable*. If one or both of $|\phi\rangle \geq |\psi\rangle$ and $|\psi\rangle \geq |\phi\rangle$ hold, then we say that the two ensembles or states are *LOCC comparable*.

If there is a path in a graph between every pair of vertices then the graph is called a *connected graph*. *EPR trees* are *connected* EPR graphs with n vertices and exactly $n - 1$ edges. A *spanning tree* is a graph which connects all vertices without forming cycles. EPR trees are indeed spanning trees. There is a unique path between any two vertices in a spanning tree. There may be more than one path between a pair of vertices in an arbitrary graph.

EC hypertrees are EC hypergraphs with no *cycles*; hypertrees have at most one vertex common between any two hyperedges. *Connectedness* for hypergraphs is defined as follows. An alternating sequence of vertices and hyperedges $\{a, E_1, v_1, E_2, v_2, \dots, E_i, v_i, E_{i+1}, \dots, E_j, b\}$ in a hypergraph $H = (S, F)$ is called a *hyperpath* from a vertex $a \in S$ to a vertex $b \in S$ if E_i and E_{i+1} have a common vertex v_i in S , for all $1 \leq i \leq j - 1$, $a \in E_1$, and $b \in E_j$, where the vertices v_i , $1 \leq i \leq j - 1$, are distinct, and the hyperedges E_i , $1 \leq i \leq j$ are distinct. If the start and end vertices of a hyperpath are identical then the hyperpath is called a *cycle* and the hypergraph is said to be *cyclic*. If the hypergraph H has a hyperpath between every pair $a, b \in S$, then H is said to be *connected*.

The *degree* of a vertex in a (hyper)graph is the number of (hyper)edges containing that vertex, in the (hyper)graph. The *degree* of a vertex subset in a hypergraph is the number of hyperedges containing all vertices of the vertex subset, in the hypergraph. We use $E(G)$ to denote the set of (hyper)edges of a (hyper)graph G .

Partial entropic arguments as applicable to ensembles of multipartite CAT states were used in the method of *bicolored merging* as in [10, 12] to establish several important LOCC incomparability results. We begin this paper by discussing the equivalence of partial entropic criteria and the method of bicolored merging in Section II, as applied to such ensembles as EPR graphs and EC hypergraphs. In Appendix A, we elaborate a formal proof of this equivalence. It was shown in [12] that (i) $n - 2$ copies of n -CAT states shared between n geographically separate parties cannot be converted to an *EPR tree* shared between the n parties using only *LOCC*, and (ii) two distinct r -uniform *EC hypertrees* shared between n geographically separated parties are *LOCC* incomparable. In this paper we further demonstrate the power of partial entropic criteria in Section III by proving the LOCC incomparability of any two distinct r -uniform *EC hypergraphs* having the same number of hyperedges, for even integers $r \geq 2$, generalizing results in [10, 12] for r -uniform EC hypertrees. The question of incomparability remains open for r -uniform EC hypergraphs with the same number of hyperedges, for odd values of $r \geq 3$. We conjecture that incomparability holds for odd values of r as well. In order to establish incomparability results we use (i) the *inclusion-exclusion principle*, and (ii) the generalized notion of *degree of a vertex subset*, to model partial entropy of *reduced or collapsed hypergraphs*. Using the same technique, we present a significantly simpler proof of the LOCC incomparability result in [12] of distinct r -uniform EC hypertrees for all integers $r \geq 2$. We observe that changing the set of hyperedges but keeping the total amount of entanglement (in the sense of the number of CAT states or hyperedges in an ensemble) fixed, induces incomparability in these distinct EC hypergraphs. Changing the set of hyperedges in this manner, we can generate numerous mutually LOCC incomparable hypergraphs in a natural *partial order* called the *LOCC partial order*, defined as follows. A node in this *LOCC partial order* represents an equivalence class of states that are mutually *LU (locally unitary) equivalent*. The directed edge (v, w) exists in the *directed acyclic graph* representing this partial order, if any state in the equivalence class of v can be transformed by LOCC to a state in the equivalence class w . In Section IV, we obtain the maximum number of mutually LOCC incomparable r -uniform EC hypergraphs, using *Sperner's Theorem* [4] for even r . This yields the *width* of the LOCC partial order, which is exponential in n . This demonstrates the necessity of quantum communication for transformations between these numerous incomparable states. These results are reported in [11].

In this paper we also study incomparability for ensembles where partial entropy criteria are not useful in establishing incomparability. The well known example of 3EPR-2GHZ falls in this category (see [5]). In Section III, we provide another example of a pair of ensembles represented by 3-uniform hypergraphs having 4 hyperedges each, which cannot be shown to be incomparable using partial entropic criteria. The EC hypergraphs H_1 and H_2 representing these ensembles have hyperedge sets $E_1 = \{\{123\}, \{156\}, \{245\}, \{346\}\}$ and $E_2 = \{\{456\}, \{234\}, \{136\}, \{125\}\}$, respectively. We establish the incomparability of these hypergraphs using LU inequivalence of reduced states following results in [5]. As conjectured above, we believe that any two r -uniform hypergraphs with equal number of hyperedges are LOCC incomparable for all integers $r \geq 2$.

Finally, using combinatorial techniques in Section V, we characterize LOCC comparability for EPR graphs in a model where LOCC is restricted to the operations of *edge destruction* and *teleportation*. The NP-completeness of the problem of deciding LOCC comparability in this restricted model follows from results in [13]. We also show that restricted LOCC is equivalent to a model of LOCC where new edges are added one at a time.

II. PARTIAL ENTROPY, BICOLORED MERGING AND LOCC INCOMPARABILITY

Suppose we create a bipartition amongst the n parties in such a way that the partial entropy is different for the two given states, $|\psi\rangle$ and $|\phi\rangle$, where these states are represented by EPR graphs or EC hypergraphs. In the case of EPR graphs, the difference in partial entropy between the two states is simply the difference between the number of EPR pairs shared across the partition in the two states. In the case of multipartite states represented by EC hypergraphs, the difference in partial entropies is equal to the difference in the number of multipartite CAT states shared across the partition in the two states. In both these cases, the state corresponding to the higher entropy cannot be obtained from that with lower entropy, as long as only *LOCC* is used.

In order to show that a multipartite state $|\psi\rangle$ can not be converted to the multipartite state $|\phi\rangle$ by *LOCC*, we may partition the original set of parties into (only) two hypothetical *merged parties or entities* say, A and B . We may also view this as coloring the parties with two colors, one for those assigned to A and the other for those to B . Parties in set A are merged into one single party. Similarly, parties in set B are merged into another single party. Each hyperedge shared between parties of A and B in $|\psi\rangle$ (or $|\phi\rangle$) is reduced to a single hypothetical EPR pair between the merged parties A and B resulting in the *bicolor merged graph (BCM)* say H_1^{bcm} (or H_2^{bcm}), as defined in [12]. Then we count the number of (such hypothetical) EPR-pairs shared across the merged parties A and B in these two graphs. If H_1^{bcm} has a smaller such count then $|\psi\rangle$ cannot be transformed by LOCC into the other state $|\phi\rangle$ whose BCM H_2^{bcm} has a larger count. The partitioning into two parts and the collapsing of all parties into these two parts is referred to as *bicolored merging* in [12]. See Figures 1 and 2 for an illustration.

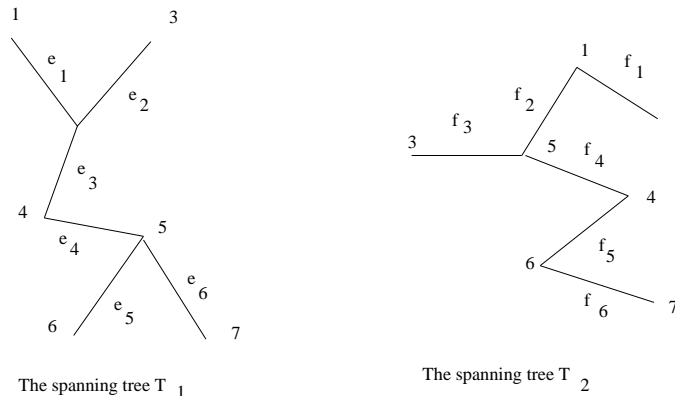


FIG. 1: Two LOCC incomparable 7-vertex EPR spanning trees (2-uniform hypertrees) from [12].

Now we formally state the scope of the technique of *bicolored merging* in establishing LOCC incomparability of EPR graphs and EC hypergraphs in graph theoretic notation and terminology as follows. Let H_1 and H_2 be two *EC hypergraphs* shared between n geographically separated parties such that a bipartition (A, B) of set of n vertices has strictly smaller partial entropy for H_1 . So H_1 cannot be transformed to H_2 by LOCC. The partition (A, B) of the set of parties may be viewed as a *cut* of the hypergraph, cutting across hyperedges that have at least one vertex in each of the parts A and B . The number of hyperedges of H_1 (H_2) across the cut is called the *capacity* of the cut in the respective hypergraph.

Observation 1. *If the capacity of a cut (A, B) is strictly smaller in EC hypergraph H_1 than in EC hypergraph H_2 , then H_1 cannot be transformed into H_2 by LOCC.*

Bicolored merging, or equivalently, the method of partial entropy may not help in establishing the LOCC incomparability of certain pairs of states. The example of the two states viz., 3EPRs and 2GHZs, shared between three parties as in [5], is an example where partial entropic methods cannot help us in establishing their LOCC incomparability.

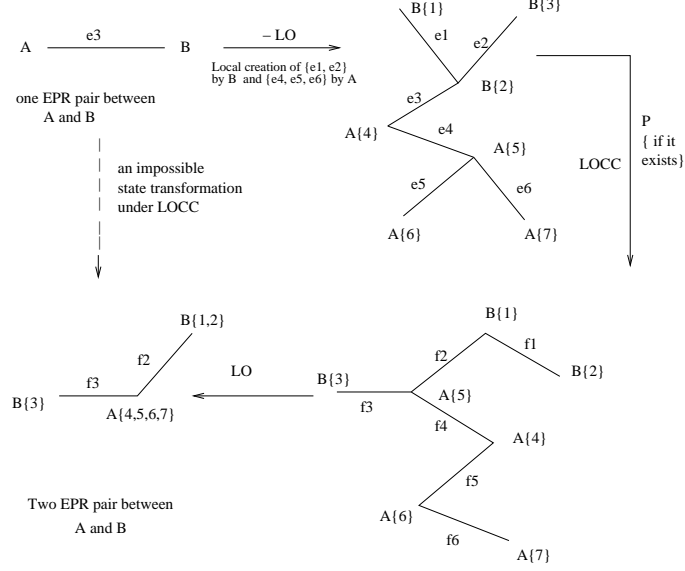


FIG. 2: Illustration of the method of *bicolored merging*, establishing LOCC incomparability as in [12].

III. COMBINATORICS OF ENTANGLEMENT CONFIGURATION HYPERGRAPHS (EC HYPERGRAPHS)

LOCC operations can transform one EC hypergraph into another. There are examples of large sets of EC hypergraphs that are mutually LOCC incomparable. One such set is that of labeled r -uniform hypertrees [12]. We first establish results for EPR graphs and then generalize them to certain classes of EC hypergraphs. The following lemma applies to EC hypergraphs and EPR graphs; the proof is presented for the general case of EC hypergraphs.

Lemma 1. *The degree of a vertex v in an EC hypergraph (or in an EPR graph) cannot increase under LOCC transformations.*

Proof. Let H_1 be a EC hypergraph which can be transformed into another EC hypergraph H_2 by LOCC. For a vertex $v \in H_1$, define a bipartition of H_1 by placing v in one set and the remaining vertices in the other. The number of edges across the cut defined by this bipartition is equal to the degree of v . By the contrapositive of Observation 1 above, it follows that the degree of v cannot increase under LOCC. \square

The above lemma is of great importance as it provides a localized view of a party and states that its total entanglement measure with other parties does not go up under LOCC operations. Using this result we generalize the incomparability result of EPR trees as in [12] to EPR graphs with the same number of edges.

Theorem 1. *Any two distinct labeled EPR graphs with the same number of vertices and edges are LOCC incomparable.*

Proof. Let G and H be two distinct labeled EPR graphs defined on the same set V of vertices, such that both the graphs have the same number of edges. For the sake of contradiction, suppose G and H are not LOCC incomparable. Then, by the definition of LOCC incomparability, either $G \geq H$ or $H \geq G$. Without loss of generality, it can be assumed that $G \geq H$ (i.e., G can be transformed to H using LOCC). Since both graphs have the same number $E(G) = E(H)$ of edges,

$$\sum_{v \in V} \deg_G(v) = 2|E(G)| = 2|E(H)| = \sum_{v \in V} \deg_H(v) \quad (1)$$

where $\deg_G(v)$ ($\deg_H(v)$) is the degree of the vertex $v \in V$ in EPR graph G (H). Further, by Lemma 1, the degree of no vertex can increase under LOCC. Therefore, the degrees of all vertices remain unchanged. Since the two graphs H and G are distinct, there exists an edge $\{u, v\}$ in G , which is not present in H . Define a bipartition $(\{u, v\}, V \setminus \{u, v\})$ of the graph G by coloring vertices u and v with color 1, and the rest of the vertices with color 2. See Figure 3. The number of edges across the cut in this partition is $\deg_G(u) + \deg_G(v) - 2$. Since (u, v) is not present in H , the same cut due to the same bipartition of the vertices will have $\deg_H(v) + \deg_H(u)$ edges in H . Since the degree of

each labeled vertex is the same in both G and H , the number of edges in the reduced graph after bicolored merging increases by 2. This is not possible under LOCC by Observation 1. So, contrary to our assumption, the two labeled EPR graphs G and H must be identical. \square

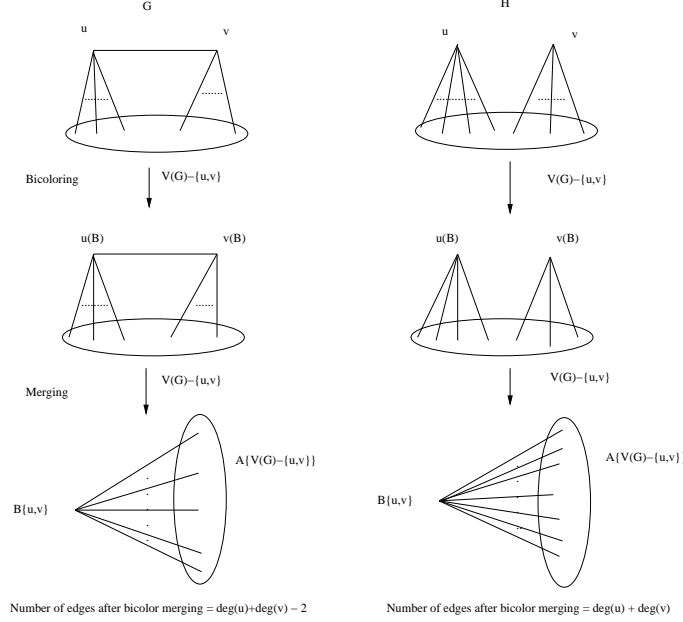


FIG. 3: LOCC incomparability of two EPR graphs with equal number of vertices and edges.

EPR graphs can be viewed as 2-uniform EC hypergraphs. It is natural to expect that results shown above for EPR graphs hold also for r -uniform EC hypergraphs for $r \geq 3$.

Lemma 2. *Let H_1 and H_2 be two r -uniform EC hypergraphs defined on the same vertex set V . If H_1 and H_2 have the same number of hyperedges and $H_1 \geq H_2$, then the degrees of all vertices in H_1 and H_2 are the same.*

Proof. The degree of a vertex can not increase under LOCC (see Lemma 1). Also, H_1 and H_2 have same number of hyperedges. Therefore the sum of degrees of all vertices in H_1 is equal to the sum of the degrees of all vertices in H_2 . This enforces the degrees of all vertices to be same for hypergraphs H_1 to H_2 . \square

In order to generalize the result as in Theorem 1 to r -uniform EC hypergraphs, we define the *degree of a vertex set*, generalizing the notion of degree of a vertex.

Definition 1. *For an EC hypergraph H with vertex set V and subset $S \subseteq V$, $\deg_H(S)$ is defined as the number of hyperedges in H containing all the vertices of S .*

Observe that $\deg_H(S)$ is the (usual) degree of the vertex v in H , when $S = \{v\}$. The following lemma helps us in determining partial entropies, or equivalently, capacities of cuts across bipartitions.

Lemma 3. *For a subset S of a r -uniform hypergraph H , the number of hyperedges across the cut $(S, V \setminus S)$ is given by*

$$\sum_{F \subseteq S} (-1)^{|F|-1} \deg_H(F) - \sum_{F \subseteq S, |F|=r} \deg_H(F)$$

Proof. Let E be a hyperedge intersecting S in $t \leq r$ vertices. E contributes to the first part of the sum above through the terms $(-1)^{|F|-1} \deg_H(F)$, $\forall F \subseteq E \cap S$, by virtue of the *inclusion-exclusion principle*. The contribution equals (i) 1, for the t singleton subsets $F \subseteq E \cap S$ with $|F| = 1$, (ii) -1 , for the $\binom{t}{2}$ subsets $F \subseteq E \cap S$ with $|F| = 2$, and so on, ending with $(-1)^{t-1}$, for the subset $F = E \cap S$. The total contribution of E to the first part of the sum is

$\sum_{i=1}^t (-1)^{i-1} \binom{t}{i} = 1$. The second part of the sum counts the number of hyperedges having all vertices in S . Hyperedge E belongs to the cut $(S, V \setminus S)$ if and only if $0 < t < r$. In this case E contributes $+1$ to the first part and 0 to the second part of the sum, making a net contribution of 1 . For $t = r$ (and $t = 0$) the contribution of E to both parts of the above sum is 1 (and 0) respectively, making a net contribution of 0 . Therefore, $\sum_{F \subseteq S} (-1)^{|F|-1} \deg_H(F) - \sum_{F \subseteq S, |F|=r} \deg_H(F)$ equals the number of hyperedges across the cut $(S, V \setminus S)$. \square

We now proceed with the proof of the main LOCC incomparability result for r -uniform EC hypergraphs using partial entropic criteria, for even integers $r \geq 4$. Later, we present an example of two LOCC incomparable 3-uniform EC hypergraphs with equal number of hyperedges, where the partial entropic criteria do not help in deciding incomparability.

Theorem 2. *Let H_1 and H_2 be any two labeled r -uniform EC hypergraphs defined on the set V of vertices. If H_1 and H_2 have the same number of hyperedges and either $H_1 \leq H_2$ or $H_2 \leq H_1$ then (i) $\deg_{H_1}(S) = \deg_{H_2}(S)$, $\forall S \subseteq V$ such that $|S| < r$, for all integers $r \geq 3$, and (ii) $H_1 = H_2$, for all even integers $r \geq 4$.*

Proof. We assume without loss of generality that H_1 is LOCC transformable to H_2 . It is sufficient to establish the following two claims.

Claim (i): $\forall S \subseteq V$ such that $|S| < r$, $\deg_{H_1}(S) = \deg_{H_2}(S)$, for all integers $r \geq 3$.

Claim (ii): $\forall S \subseteq V$ such that $|S| = r$, $\deg_{H_1}(S) = \deg_{H_2}(S)$, only for even integers $r \geq 4$. [For sets $S \subseteq V$ with $|S| = r$, $\deg_{H_i}(S) = 1$ if there is a hyperedge in H_i consisting of vertices in S , and $\deg_{H_i}(S) = 0$, otherwise. Therefore, establishing the equality of $\deg_{H_1}(S)$ and $\deg_{H_2}(S)$ for all subsets $S \subseteq V$ with $|S| = r$ implies that the two hypergraphs are identical.] Claim (i) is established by induction on the cardinality k of the $S \subseteq V$; Claim (ii) is subsequently established for even integers $r \geq 4$.

Proof of Claim (i):

Basis step: For $k = 1$ the claim holds by Lemma 2; if two r -uniform EC hypergraphs with the same number of hyperedges and vertices are not incomparable then they have the same vertex degrees. In other words, $\deg_{H_1}(S) = \deg_{H_2}(S)$, where $S = \{v\}$, $\forall v \in V$.

Induction hypothesis: Assume that $\deg_{H_1}(S) = \deg_{H_2}(S)$, for all $S \subseteq V$ such that $|S| = k$, for all $k \leq m < r - 1$.

Induction Step: We require to show that $\deg_{H_1}(S) = \deg_{H_2}(S)$, for all $S \subseteq V$ such that $|S| = m + 1$.

Clearly, no hyperedge in either hypergraph can have all its vertices in $S \subseteq V$ as $|S| = k \leq m + 1 < r$. Therefore by Lemma 3, the number of edges across the cut $(S, V \setminus S)$ in H_i , $i = \{1, 2\}$, is given by:

$$\sum_{F \subseteq S} (-1)^{|F|-1} \deg_{H_i}(F) \quad (2)$$

Since the cut capacity cannot increase under LOCC it follows that

$$\sum_{F \subseteq S} (-1)^{|F|-1} \deg_{H_1}(F) \geq \sum_{F \subseteq S} (-1)^{|F|-1} \deg_{H_2}(F) \quad (3)$$

By the induction hypothesis, $\deg_{H_1}(F) = \deg_{H_2}(F)$, for all $F \subseteq V$ with $|F| \leq m$. Canceling out subsets which contribute equally to both sides of equation (3) we have

$$(-1)^{|S|-1} \deg_{H_1}(S) \geq (-1)^{|S|-1} \deg_{H_2}(S). \quad (4)$$

for all $S \subset V$ with $|S| = m + 1$. Since the hypergraphs are r -uniform,

$$\sum_{S \subseteq V, |S|=(m+1)} \deg_{H_i}(S) = \binom{r}{m+1} |E(H_i)|, i \in \{1, 2\} \quad (5)$$

where $E(H_1) = E(H_2)$ is the number of hyperedges in each EC hypergraph. Given a hyperedge, $\binom{r}{m+1}$ subsets of the vertices in that hyperedge would contribute 1 to the sum of the left hand side. So, the total contribution over all hyperedges is the number of hyperedges in H_i times $\binom{r}{m+1}$. Summing up relation (4) over all $S \subseteq V$,

$$\sum_{S \subseteq V, |S|=(m+1)} (-1)^m \deg_{H_1}(S) \geq \sum_{S \subseteq V, |S|=(m+1)} (-1)^m \deg_{H_2}(S)$$

By equation (5) both the sums are equal and therefore equality holds in the relation (4), for all $S \subseteq V$ with $|S| = m+1$. Therefore,

$$\deg_{H_1}(S) = \deg_{H_2}(S) \quad (6)$$

for all $S \subseteq V$ with $|S| = m+1 < r$. Claim (i) of this theorem therefore holds by induction.

Proof of Claim (ii):

Here we have $S \subseteq V$, such that $|S| = r$. By Lemma 3, the number of edges across the cut $(S, V \setminus S)$ is given by:

$$\left(\sum_{F \subseteq S} (-1)^{|F|-1} \deg_{H_i}(F) \right) - \deg_{H_i}(S) \quad (7)$$

Since the cut capacity does not increase under LOCC, like inequality (3), the following inequality holds

$$\left(\sum_{F \subseteq S} (-1)^{|F|-1} \deg_{H_1}(F) \right) - \deg_{H_1}(S) \geq \left(\sum_{F \subseteq S} (-1)^{|F|-1} \deg_{H_2}(F) \right) - \deg_{H_2}(S) \quad (8)$$

By virtue of the already established Claim (i) above, $\deg_{H_1}(F) = \deg_{H_2}(F)$, for all $F \subseteq V$ with $|F| \leq m \leq r-1$. Canceling out subsets which contribute equally to both sides of equation (8), we get

$$2(-1)^{|S|-1} \deg_{H_1}(S) \geq 2(-1)^{|S|-1} \deg_{H_2}(S) \quad (9)$$

for even integers r . The multiple 2 appears since the last (negative) term $(-1)^{r-1} \deg_{H_i}(S)$ in the summation on each side of the inequality 8 adds up with the negative term $-\deg_{H_i}(S)$, for even integers $r \geq 4$. Using remaining arguments as in the proof of Claim (i), and the equation (9), we have

$$\deg_{H_1}(S) = \deg_{H_2}(S) \quad (10)$$

for all $S \subseteq V$, $|S| = r$, only for even $r \geq 4$, thereby establishing Claim (ii). \square

For an r -uniform hypergraph H with r odd, the hyperedges having all vertices in S contribute $(-1)^{r-1} = 1$ to the first term and (-1) to the second term in Lemma 3, and therefore cancel out. Therefore, the number of hyperedges across a cut $(S, V \setminus S)$ is given by

$$\sum_{F \subseteq S, |F| < r} (-1)^{|F|-1} \deg_H(F)$$

For all cuts, cut capacities are determined by the quantities $\deg_H(F)$ for $F \subset V$, $|F| < r$. We now exhibit two 3-uniform hypergraphs on 6 vertices having 4 edges such that all cut capacities are same in both the hypergraphs.

$$H_1 = \{123\}, \{156\}, \{245\}, \{346\}$$

$$H_2 = \{456\}, \{234\}, \{136\}, \{125\}$$

It is easy to verify that $\deg_{H_1}(F) = \deg_{H_2}(F)$ for $F \subset V$ and $|F| < 3$. From the above argument it follows that H_1 and H_2 cannot be shown to be incomparable by partial entropic characterizations.

As $\deg_{H_1}(F) = \deg_{H_2}(F)$ for $F \subset V$ and $|F| < 3$ so they are isentropic [5]. And from [5] that isentropic states are either LU (locally unitary) equivalent or incomparable. Partition the vertices into three sets $A = \{1\}, B = \{2, 3\}, C = \{4, 5, 6\}$ and merge the vertices in the same sets. H_1 reduces to EPR graph with edges $(A, B), (A, C)$ and two copies of (B, C) . H_2 reduces to EC graph with 2 GHZ shared between A, B, C and an EPR pair shared between B and C . Let the reduced graph of H_1 and H_2 be denoted by $R(H_1)$ and $R(H_2)$ respectively. If H_1 and H_2 are LU equivalent then so is $R(H_1)$ and $R(H_2)$. To prove that they are not LU equivalent, observe that the mixed state obtained by tracing out B from $R(H_2)$ state i.e., $\rho_{AC}(R(H_2))$ is a maximally mixed, separable state of A and C , while the corresponding mixed state $\rho_{AC}(R(H_1))$ obtained from $R(H_1)$ can be distilled to entangled state, consisting on intact (A, C) EPR pair shared by the two parties. So if $R(H_1)$ and $R(H_2)$ are LU equivalent, then A and C can do local unitary operations and convert $\rho_{AC}(R(H_2))$ to $\rho_{AC}(R(H_1))$. This is not possible as one cannot make entanglement by LOCC, $R(H_2)$ and $R(H_1)$ are not LU equivalent then H_1 and H_2 are not LU equivalent, therefore by [5] they are LOCC incomparable.

Next we propose an alternative proof of the incomparability result of [12] about distinct r -uniform EC hypertrees as follows.

Theorem 3. *Any two distinct ensembles of multipartite CAT states represented by r -uniform EC hypertrees are LOCC incomparable.*

Proof. Let the two hypertrees defined on the vertex set $\{1, 2, \dots, n\}$ be T_1 and T_2 . For $r = 2$ the result follows from Theorem 1 as all trees on n vertices have $n - 1$ edges.

For $r = 3$, we assume without loss of generality that the hyperedge $\{1, 2, 3\}$ is in $T_1 \setminus T_2$. If T_1 and T_2 are not LOCC incomparable, then by part (i) in Theorem 2, we have $\deg_{T_1}(\{1, 2\}) = \deg_{T_2}(\{1, 2\})$. So, there should be a hyperedge $E_1 = \{1, 2, x\}$ in $T_2 \setminus T_1$, where x is not in $\{1, 2, 3\}$; this hyperedge E_1 cannot be in T_1 because no hypertree has two hyperedges with two common vertices. Similarly, T_2 must have hyperedges $E_2 = \{1, 3, y\}$ and $E_3 = \{2, 3, z\}$, where y is not in $\{1, 2, 3, x\}$ and z is not in $\{1, 2, 3, x, y\}$. This implies that the cycle $\{3, E_2, 1, E_1, 2, E_3, 3\}$ is present in T_2 , a contradiction.

For $r > 3$, we assume without loss of generality that the hyperedge $\{1, 2, \dots, r\}$ is in $T_1 \setminus T_2$. If T_1 and T_2 are not LOCC incomparable, then by part (i) of Theorem 2, we have $\deg_{T_1}(\{1, 2, \dots, r-1\}) = \deg_{T_2}(\{1, 2, \dots, r-1\})$ and $\deg_{T_1}(\{2, 3, \dots, r\}) = \deg_{T_2}(\{2, 3, \dots, r\})$. So, there must be hyperedges in T_2 containing $\{1, 2, \dots, r-1\}$ and $\{2, 3, \dots, r\}$. As $r > 3$, these two hyperedges intersect in at least 2 vertices, a contradiction. \square

IV. PARTIAL ORDER INDUCED BY LOCC AND ITS WIDTH

In this section we define a *partial order* where each node represents an equivalence class of states that are mutually LU equivalent. This partial order is called the *LOCC partial order* and is represented by a *directed acyclic graph* G_{LOCC} . We define this directed acyclic graph $G_{LOCC}(V, E)$ as follows. Each vertex or node $v \in V$ represents an equivalence class of states that are mutually LU equivalent. The directed edge $(v, w) \in E$ (directed from v to w) exists in $G_{LOCC}(V, E)$ if any state in the equivalence class of $v \in V$ can be transformed by LOCC to a state in the equivalence class $w \in V$. [By this definition of a directed edge, there is a self loop in every node.] We denote this partial order by the (binary) relation \geq_{LOCC} between the nodes of the directed graph G_{LOCC} ; for $X, Y \in V$, $X \geq_{LOCC} Y$, if and only if there is a directed edge from X to Y in G_{LOCC} .

Lemma 4. *The directed graph $G_{LOCC}(V, E)$ representing the LOCC partial order is a transitive graph i.e., if there is a directed edge $(v, w) \in E$, and a directed edge $(w, z) \in E$, then there is also a directed edge $(v, z) \in E$.*

Proof. An edge from v to w implies that there exists an LOCC protocol transforming a state $s \in v$ to an state $t \in w$. The directed edge (w, z) implies that exists another LOCC protocol which converts the state $t \in w$ to a state $u \in z$. Applying the two protocols in succession, s can be converted to u , enforcing the directed edge (w, z) . \square

Corollary 1. *The graph $G_{LOCC}(V, E)$ has no non-trivial cycles.*

It can be shown that multipartite quantum states form a partial order under LOCC transformations (see [14]). Further the LOCC equivalent classes of quantum states also form a partial order under the relation \geq_{LOCC} as defined above. The relation \geq_{LOCC} is a partial order as it satisfies the three properties:

1. The relation is reflexive since for all nodes of G_{LOCC} $X \geq_{LOCC} X$; each node of G_{LOCC} has a self loop.
2. The relation is transitive, i.e., if $X \geq_{LOCC} Y$ and $Y \geq_{LOCC} Z$ then $X \geq_{LOCC} Z$, as already shown earlier in Lemma 4.
3. The relation is antisymmetric. Since if $X \geq_{LOCC} Y$ and $Y \geq_{LOCC} X$, then X is identical to Y since there cannot be cycles in G_{LOCC} except for self loops, as shown earlier.

Lemma 5. *The relation \geq_{LOCC} among the nodes of the graph G_{LOCC} forms a partial order.*

The *width* of \geq_{LOCC} is the maximum number of mutually LOCC incomparable EC hypergraphs in \geq_{LOCC} . The width of \geq_{LOCC} can be obtained using Theorem 2 and Sperner's Theorem [4].

Sperner's Theorem. *The maximum cardinality of a collection of subsets of a n element set, none of which contains another is $\binom{n}{\lfloor n/2 \rfloor}$.*

Now we provide the derivation of the *width* of the partial order for r -uniform EC hypergraphs using Sperner's Theorem, where r is an even integer.

Theorem 4. *The maximum number of mutually LOCC incomparable r -uniform EC hypergraphs (for even r), with n nodes is $\binom{M}{\lfloor M/2 \rfloor}$, where $M = \binom{n}{r}$.*

Proof. Let $r > 3$ be an even number. The maximum number of hyperedges possible in an r -uniform EC hypergraph with n vertices is $M = \binom{n}{r}$. Let S be any set of $N > \binom{M}{\lfloor M/2 \rfloor}$ mutually LOCC incomparable distinct r -uniform EC hypergraphs defined on n vertices. By Sperner's Theorem, there must be two hypergraphs H_1 and H_2 in the collection S such that the set of hyperedges in H_1 is a subset of the set of hyperedges in H_2 . This contradicts the assumption that H_1 and H_2 are LOCC incomparable because $H_1 \geq_{LOCC} H_2$ by a simple LOCC transformation that drops all additional hyperedges in $H_1 \setminus H_2$ from H_1 . So, we know that $N \leq \binom{M}{\lfloor M/2 \rfloor}$, is an upper bound on the width of the partial order \geq_{LOCC} .

Now we show that this bound is tightly achievable as follows. Consider the set of all the different r -uniform EC hypergraphs on n vertices with exactly a fixed number $\lfloor \frac{M}{2} \rfloor$ of hyperedges. By Theorem 2, all these EC hypergraphs are LOCC incomparable. This set has $\binom{M}{\lfloor M/2 \rfloor}$ r -uniform EC hypergraphs, forming an *antichain* of the partial order \geq_{LOCC} . Therefore the width of the partial order \geq_{LOCC} is $\binom{M}{\lfloor M/2 \rfloor}$ where $M = \binom{n}{r}$. \square

V. RESTRICTION OF LOCC TO TELEPORTATION AND EPR DESTRUCTION

Now we consider LOCC restricted to the two basic operations of *edge destruction* and *teleportation* on EPR graphs. The allowed operations are (i) discarding an edge (destroying an EPR pair) and (ii) teleportation, i.e., replacing edges (EPR pairs) $\{x, y\}$ and $\{y, z\}$ by one EPR pair across edge $\{x, z\}$. If EPR graphs G and H are such that H can be obtained from G by such restricted LOCC, then we say $G \geq_R H$. In the following lemma we characterize graph theoretic properties such that $G \geq_R H$, for EPR graphs G and H .

Lemma 6. *Let G and H be two EPR graphs defined on the same vertex set V . Then, $G \geq_R H$ if and only if there are edge disjoint paths in G from u to v for all edges $\{u, v\}$ in H .*

Proof. For the if part, observe that a path P from u to v in G can be reduced to the EPR edge $\{u, v\}$ using successive steps of teleportation.

$$uv_1v_2v_3 \dots v_kv \rightarrow uv_1v_3 \dots v_kv \rightarrow \dots \rightarrow uv \quad (11)$$

Therefore edge disjoint paths from u to v can be converted independently to edges $\{u, v\}$ in H . The remaining edges of G on this path are discarded to transform G to H .

For the only if part, consider the inverse of the two possible operations

1. Adding an edge
2. Replacing an edge by a path of length 2.

If $G \geq_R H$, we can create G from H using the inverse operations as follows. As we keep applying these inverse operation steps for constructing G from H , the invariant maintained is the presence of at least $|E(H)|$ edge-disjoint paths in the intermediate graphs. Initially we have one path in H from u to v for each edge $\{u, v\}$ in H . These edges themselves are the initial edge disjoint paths to begin with. Since the application of the first inverse operation does not destroy any edge, we need to consider only the second operation. During the application of the second inverse operation, one edge of a path may be destroyed; however, two edges are added to reconnect the path thereby preserving the edge disjointness property of all the relevant paths. Therefore G contains edge disjoint paths from u to v , for each $\{u, v\} \in E(H)$. \square

As shown earlier, partial entropic criteria are not applicable for establishing *LOCC* incomparability of certain kinds of multipartite states. The new criterion of the existence of edge disjoint paths in the EPR graph G for each edge in H , provides a stronger characterization of proving *LOCC* incomparability in the restricted model.

Consider two EPR graphs G and H in Figure 4. Since G has more edges than H , H cannot be transformed to G by LOCC. So, in order to show that EPR graphs G and H are LOCC incomparable, we need to only show that $G \geq H$ does not hold. Observe that using the bicolored merging technique cannot help us establish that $G \geq H$ does not hold. This is due to the fact that no bicoloring of the vertex set results in any violation of the non-increase of partial entropy as we go from G to H . Observe however that $G \geq_R H$ does not hold by Lemma 6 since H contains three new edges $\{A, B\}$, $\{B, C\}$, $\{C, A\}$, but G does not contain edge disjoint paths from A to B , B to C , and C to D . So, the two graphs are *LOCC* incomparable under our restricted *LOCC* model. A natural open question is whether the two EPR graphs G and H are LOCC incomparable in the general model. We conjecture that they are indeed LOCC incomparable.

We now investigate whether the edge disjoint path criterion is powerful enough to capture LOCC. We show that this is indeed the case when edges in $E(H) \setminus E(G)$ appear one at a time.

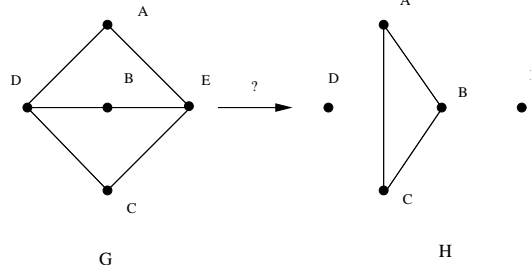


FIG. 4: Two EPR graphs incomparable under restricted LOCC, not yielding to partial entropic criteria.

Definition 2. An LOCC transformation from G to H is called good if $|E(H) \setminus E(G)| \leq 1$, where G and H are EPR graphs defined on the same vertex set V .

Lemma 7. Suppose EPR graph G can be transformed to H via a good transformation. Then, $G \geq_R H$.

Proof. If $E(H) \setminus E(G)$ is empty, we can create H from G by a sequence of EPR edge destructions. For the case where $E(H) \setminus E(G) = 1$, we present a constructive proof depicting a path from u to v in G , for the single edge $\{u, v\} \in E(H) \setminus E(G)$, where all the edges of the constructed path belong to $E(G) \setminus E(H)$. This path is sufficient to establish $G \geq_R H$.

We construct the path from u to v in $E(G) \setminus E(H)$ as follows. Initialize the vertex set $C_1 = \{u\}$ and the set of edges defined on C_1 as $E(C_1) = \phi$. Set $i = 1$. Perform the following steps until termination in Step 4.

1. Consider the cut $(C_i, V \setminus C_i)$. Since edge $\{u, v\} \in E(H) \setminus E(G)$, the capacity of this cut cannot increase under LOCC. So, there must be vertices $u'_i \in C_i$ and $w_i \in V \setminus C_i$ such that the edge $\{u'_i, w_i\}$ is in $E(G) \setminus E(H)$. Find such an edge $\{u'_i, w_i\}$.
2. $C_{i+1} = C_i \cup \{w_i\}$.
3. $E(C_{i+1}) = E(C_i) \cup \{u'_i, w_i\}$ i.e., add the edge to $E(C_i)$, yielding tree $E(C_{i+1})$ over vertex set C_{i+1} .
4. If $w_i = v$ then stop else $i = i + 1$.

The invariant at the beginning of each iteration of the above procedure is that the subgraph $G(C_i, E(C_i))$ of G with vertex set C_i and edge set $E(C_i)$ is a tree; this subgraph is connected, and has exactly $|C_i| - 1$ edges. When the process terminates, the tree $E(C_i)$ in $E(G) \setminus E(H)$ contains both u and v . So, there is a path from u to v using edges entirely from $E(G) \setminus E(H)$.

Using this path we can perform LOCC transformations in our restricted model using repeated teleportation steps, thereby creating the only new EPR edge $\{u, v\} \in E(H) \setminus E(G)$. We can destroy the remaining EPR pairs from G that do not belong to $E(H)$, finally yielding H . So, $G \geq_R H$. \square

If an EPR graph H can be obtained from an EPR graph G by a sequence of good transformations, then we know that $G \geq_R H$ by the repeated application of Lemma 7. For the converse, suppose $G \geq_R H$. Then, by Lemma 6, there are edge disjoint paths in G for all edges $\{u, v\}$ in H . In order to generate H from G , we may therefore use such disjoint paths, one at a time, to generate the edges in H that do not exist in G . Each such transformation is a good transformation since it generates at most one new EPR edge in the resulting intermediate EPR graph; using a sequence of such good transformations, G can be converted to H . We now summarize our characterizations as follows.

Theorem 5. Let G and H be EPR graphs defined on the same vertex set. The following statements are equivalent.

1. $G \geq_R H$.
2. There are edge disjoint paths in G from u to v for all edges $\{u, v\}$ in H .
3. H can be obtained from G by a sequence of good transformations.

Given two EPR graphs G and H , the decision problem of determining whether $G \geq_R H$ is NP-hard since this problem can be used to solve the decision problem of checking for edge disjoint paths in G . The problem of deciding the existence of edge disjoint paths in graphs was shown to be NP-complete in [13].

It remains open to determine whether our restricted LOCC \geq_R is powerful enough to capture (general) LOCC for EPR graphs. We believe that the two models are equally powerful for EPR graphs.

VI. CONCLUDING REMARKS

Partial entropic criteria are not sufficient for demonstrating LOCC incomparability between multipartite states. New techniques need to be developed for a better understanding of LOCC comparability. Further, it would be interesting to investigate whether the restricted model of LOCC studied in this paper (which uses only teleportation and EPR pair destruction), is powerful enough to capture LOCC in general for multipartite states comprising multiple EPR pairs. For LOCC incomparable ensembles, the amount of quantum communication necessary for transformations, and the possibility of classifications based on some notions of quantum distance between ensembles may be studied.

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APPENDIX A: BICOLORED MERGING AND PARTIAL ENTROPY

In this appendix we show the equivalence of *partial entropic criteria* and the technique of *bicolored merging* in establishing LOCC incomparability of multipartite states represented by EC hypergraphs and EPR graphs. Let A and B be disjoint sets of parties sharing the quantum state ρ^{AB} between them. If $\rho^{AB} = \rho \otimes \sigma$, where ρ is the density operator of the system A , and σ is a density operator for the system B , then we know from page 106. of [2] that the partial entropy

$$\rho^A = \text{tr}_B(\rho^{AB}) = \text{tr}_B(\rho \otimes \sigma) = \rho. \quad (\text{A1})$$

Also, if $\rho^{AB} = \rho$ where ρ is the density operator of the system A . Then the partial entropy

$$\rho^A = \text{tr}_B(\rho^{AB}) = \text{tr}_B(\rho) = \rho. \quad (\text{A2})$$

We also use the important property of *Von-Neumann entropy* from page 514 of [2] that,

$$S(\rho \otimes \sigma) = S(\rho) + S(\sigma) \quad (\text{A3})$$

Let the parties of A and B share an n -CAT state where the first r qubits is with the parties of the set A and the remaining qubits from $r + 1$ to n is with the parties of the set B . This state has the density operator

$$\rho^{AB} = \left(\frac{|0_1 \dots 0_r 0_{r+1} \dots 0_n\rangle + |1_1 \dots 1_r 1_{r+1} \dots 1_n\rangle}{\sqrt{2}} \right) \left(\frac{\langle 0_1 \dots 0_r 0_{r+1} \dots 0_n| + \langle 1_1 \dots 1_r 1_{r+1} \dots 1_n|}{\sqrt{2}} \right)$$

We denote $|0_1 \dots 0_r\rangle$ by $|0_A\rangle$, and $|1_1 \dots 1_r\rangle$ by $|1_A\rangle$. We use similar notation for the set B . Then,

$$\begin{aligned} \rho^{AB} &= \left(\frac{|0_A 0_B\rangle + |1_A 1_B\rangle}{\sqrt{2}} \right) \left(\frac{\langle 0_A 0_B| + \langle 1_A 1_B|}{\sqrt{2}} \right) \\ &= \frac{|0_A 0_B\rangle \langle 0_A 0_B| + |1_A 1_B\rangle \langle 0_A 0_B|}{2} + \frac{|0_A 0_B\rangle \langle 1_A 1_B| + |1_A 1_B\rangle \langle 1_A 1_B|}{2} \end{aligned} \quad (\text{A4})$$

Tracing out the system B from ρ_{AB} , we find the reduced density operator of the system A ,

$$\begin{aligned} \rho^A &= \text{tr}_B(\rho^{AB}) \\ &= \frac{\text{tr}_B|0_A 0_B\rangle \langle 0_A 0_B| + \text{tr}_B|1_A 1_B\rangle \langle 0_A 0_B|}{2} + \frac{\text{tr}_B|0_A 0_B\rangle \langle 1_A 1_B| + \text{tr}_B|1_A 1_B\rangle \langle 1_A 1_B|}{2} \\ &= \frac{|0_A\rangle \langle 0_A| \langle 0_B 0_B\rangle + |1_A\rangle \langle 0_A| \langle 0_B 1_B\rangle}{2} + \frac{|0_A\rangle \langle 1_A| \langle 1_B 0_B\rangle + |1_A\rangle \langle 1_A| \langle 1_B 1_B\rangle}{2} \\ &= \frac{|0_A\rangle \langle 0_A| \langle 0_B 0_B\rangle + |1_A\rangle \langle 1_A| \langle 1_B 1_B\rangle}{2} \\ &= \frac{|0_A\rangle \langle 0_A| + |1_A\rangle \langle 1_A|}{2} \end{aligned} \quad (\text{A5})$$

From A5 we get,

$$\begin{aligned} S_B(\rho^{AB}) &= -\text{tr}(\rho^A \log_2 \rho^A) \\ &= 1. \end{aligned} \quad (\text{A6})$$

So, from A6 we conclude that $S_B(\rho^{AB}) = 1$ if an n -CAT is shared by parties of both the sets. Also, from A1 and A2, we have

$$S_B(\rho^{AB}) = 0 \quad (\text{A7})$$

if all entanglements are shared only by parties within sets A and B but not across A and B .

We proceed to prove our Theorem 6. Let H denote and EC hypergraph. Let $V(H) = C \cup D$, where C and D are disjoint. We use the following notation. Let $D(H)$ (or $C(H)$) denote the set of hyperedges shared within the elements of the set D (or C). Let $CD(H)$ denote the set of hyperedges shared across the sets C and D . We say $H_1 \not\geq H_2$ if $H_1 \geq H_2$ does not hold.

Theorem 6. *Let H_1 and H_2 be two entanglement hypergraphs shared between the geographically separated parties p_1, p_2, \dots, p_n . If it can be shown using partial entropic criteria that $H_1 \not\geq H_2$, then there exists a bicolored merging scheme establishing $H_1 \not\geq H_2$.*

Proof. Suppose there is a subset X of p_1, p_2, \dots, p_n such that $S_X(H_1) < S_X(H_2)$, thereby ensuring that H_1 cannot be transformed to H_2 using LOCC. Since each hyperedge corresponds to a GHZ state in the EC hypergraph, we denote the corresponding maximal entanglement associated with the hyperedge e as $|e\rangle$. Let the hyperedges of H_1 be $e_{11}, e_{12}, \dots, e_{1r}, \dots$, and those belonging to H_2 be $e_{21}, e_{22}, \dots, e_{2j}, \dots$. For H_1 ,

$$\rho^{H_1} = \bigotimes_{e_{1i} \in E(H_1)} |e_{1i}\rangle \langle e_{1i}| \quad (\text{A8})$$

Therefore,

$$\begin{aligned} \rho_X^{H_1} &= \text{tr}_X \bigotimes_{e_{1i} \in E(H_1)} |e_{1i}\rangle \langle e_{1i}| \\ &= \bigotimes_{e_{1i} \in E(H_1)} \text{tr}_X |e_{1i}\rangle \langle e_{1i}| \\ &= \left(\bigotimes_{e_{1s} \in X(H_1)} \text{tr}_X |e_{1s}\rangle \langle e_{1s}| \right) \otimes \left(\bigotimes_{e_{1t} \in \bar{X}(H_1)} \text{tr}_X |e_{1t}\rangle \langle e_{1t}| \right) \otimes \left(\bigotimes_{e_{1u} \in X\bar{X}(H_1)} \text{tr}_X |e_{1u}\rangle \langle e_{1u}| \right) \end{aligned} \quad (\text{A9})$$

From A1 we get,

$$\rho_X^{H_1} = \left(\bigotimes_{e_{1t} \in \bar{X}(H_1)} tr_X |e_{1t}\rangle\langle e_{1t}| \right) \otimes \left(\bigotimes_{e_{1u} \in X\bar{X}(H_1)} tr_X |e_{1u}\rangle\langle e_{1u}| \right) \quad (\text{A10})$$

From A2 we get,

$$\rho_X^{H_1} = \left(\bigotimes_{e_{1t} \in \bar{X}(H_1)} |e_{1t}\rangle\langle e_{1t}| \right) \otimes \left(\bigotimes_{e_{1u} \in X\bar{X}(H_1)} tr_X |e_{1u}\rangle\langle e_{1u}| \right) \quad (\text{A11})$$

We know that $S_{\bar{X}}(H_1) = S(\rho_X^{H_1})$. From equations A3 and A7 we get,

$$\begin{aligned} S_{\bar{X}}(H_1) &= S \left(\bigotimes_{e_{1t} \in \bar{X}(H_1)} |e_{1t}\rangle\langle e_{1t}| \right) + S \left(\bigotimes_{e_{1u} \in X\bar{X}(H_1)} tr_X |e_{1u}\rangle\langle e_{1u}| \right) \\ &= S \left(\bigotimes_{e_{1u} \in X\bar{X}(H_1)} tr_X |e_{1u}\rangle\langle e_{1u}| \right) \end{aligned} \quad (\text{A12})$$

From A3 we get,

$$S_{\bar{X}}(H_1) = \sum_{e_{1u} \in X\bar{X}(H_1)} S(tr_X |e_{1u}\rangle\langle e_{1u}|)$$

We know that,

$$S(tr_X |e_{1u}\rangle\langle e_{1u}|) = 1, \quad \forall |e_{1u}\rangle \in X\bar{X}(H_1) \quad (\text{A13})$$

From A13,

$$S_{\bar{X}}(H_1) = \text{number of hyperedges containing at least one party of } X \text{ and } \bar{X} = \{p_1, p_2, \dots, p_n\} \setminus X \quad (\text{A14})$$

Similarly for H_2 ,

$$\begin{aligned} S_{\bar{X}}(H_2) &= S(\rho_X^{H_2}) \\ &= -tr \rho_X^{H_2} \log_2 \rho_X^{H_2} \\ &= \text{number of hyperedges containing at least one party of } X \text{ and } \bar{X} = \{p_1, p_2, \dots, p_n\} \setminus X \end{aligned} \quad (\text{A15})$$

Now we use the bicolored merging scheme. We color all the vertices in the set X with one color and collapse them into the merged party A ; we color the rest of the vertices with another color and collapse them into another merged party B . The number of (hypothetical) EPR pairs in the reduced bicolor-merged graph (BCM graph) H_1^{bcm} (or H_2^{bcm}) after bicolored merging, is equal to number of hyperedges containing elements of both X and $\{p_1, p_2, \dots, p_n\} \setminus X$, which is equal to $S_X(H_1)$ (or $S_X(H_2)$) from equation A14 (A15). Since $S_X(H_1) < S_X(H_2)$, the number of EPR pairs in H_1^{bcm} is less than those present in H_2^{bcm} . So, H_1 cannot be transformed to H_2 using LOCC. \square

The number of edges in the reduced graph obtained after bicolored merging is equal to the capacity of the corresponding cut. The principle of bicolored merging can therefore be restated as follows:

Lemma 8. *Suppose G and H are EC hypergraphs such that $G \geq H$. Then the cut capacity across a cut in H cannot be greater than the cut capacity across the same cut in G .*